Cross Product in Finite-Dimensional Euclidean Spaces

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To
The Inhabitants of Space in General
And H. C. in Particular
This Work is Dedicated
By a Humble Native of Flatland
In the Hope that
Even as he was Initiated into the Mysteries
Of Three Dimensions
Having been previously conversant
With Only Two
So the Citizens of that Celestial Region
May aspire yet higher and higher
To the Secrets of Four Five or even Six Dimensions
Thereby contributing
To the Enlightenment of the Imagination
And the possible Development
Of that most rare and excellent Gift of Modesty
Among the Superior Races
Of Solid Humanity

from the dedication of *Flatland: A Romance of Many Dimensions* by Edwin A. Abbott

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Abstract

This paper presents a thorough analysis of vector cross products in finite dimensional Euclidean vector spaces. The most common case in $\Re^3$ is examined, and the cross product expression is derived separately for two and four dimensions. The most general task of defining a cross product in n-dimensions is tackled. Some applications are also presented.

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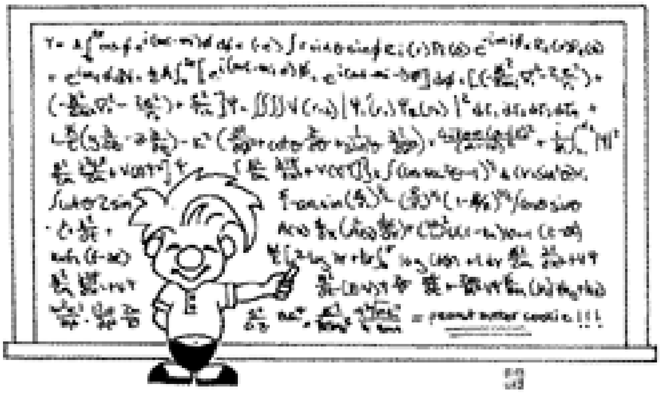
Peanut butter cookie recipe



1 Introduction

Every student of engineering and mathematics is familiar with the definition of the cross product in three dimensions. The cross product in $\Re^3$ is defined geometrically as the vector whose direction is perpendicular to two given vectors, with the proper sense determined by the right-hand rule. There exist many applications of the vector cross product in physics; including the analysis of rotational motion, and Maxwell's electromagnetic theory. The mathematics of the cross product itself is highly fascinating, and the natural question soon arises as how to extend the definition of the cross product to higher (and lower) dimensions. In this paper, we extend the cross product to n-dimensional Euclidean space. We start out with a review of the definition of the cross product in 3-space (§2). Next, we treat the cross product in two dimensions (§3), which gives us new insight into the problem. Then, we analyze Euclidean 4-space, and arrive at a definition of the 4-dimensional cross product (§4). Finally, using the insights gained in two and four dimensions, the problem of defining the cross product in n-dimensional Euclidean space is tackled (§5). This paper concludes with a brief look at some of the applications of the cross product (§6). The mathematical extension into higher dimensions of the quite familiar mathematical object of the cross product is both intriguing and beautiful.

Before we can begin to deal with cross products of vectors, we need a suitable definition of what we mean by a 'vector'. We also need to define vector addition, and scalar multiplication; we will also need to know how to take the dot product of two vectors. These are all standard operations on Euclidean $\Re^n$. We also need to be familiar with the more general concept of a vector space, which is a natural generalization of $\Re^n$. In addition, we need to be comfortable with inner product spaces, which are vector spaces with an operation on pairs of vectors called an "inner product." The inner product is a generalization of the dot product in $\Re^n$. These are all standard constructs treated quite extensively in the field of mathematics called linear algebra. For an in-depth coverage of this material, including complete definitions and examples of vector spaces and inner product spaces, please refer to Larson and Edwards [5], or, on a higher level, to Hoffman and Kunze [4]. We will assume you have read and understood the material contained in those references; we will not indulge in presenting those details here. For a somewhat different approach to the cross product, please see Massey [6] and Walsh [8]. A forthcoming paper by the author will extend the definition of the cross product to *all* finite-dimensional vector spaces, including *non*-Euclidean spaces.

2 Starting in the Middle – Cross Product in 3-Space

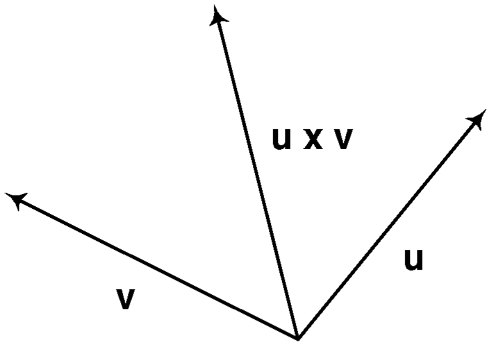
Most are already very familiar with the 3-dimensional vector cross product from studying physics or analytic geometry. We will review this material here with an eye on how the methods first introduced in $\Re^3$ can be generalized to $\Re^n$. We begin with a geometric definition of the cross product, and then derive its algebraic properties. Most textbooks do it the other way — by first defining the cross product algebraically and then deriving its geometric properties. The approach taken here is more intuitive — as the motivation for the cross product is purely geometric. In later discussions, we will be forced to start with an algebraic definition because of the difficulty in visualizing n-dimensional space. For an algebra-first approach to the cross product in $\Re^3$, see Edwards and Penney [3, pp. 725–733].

Definition 1 [Cross product in $\Re^3$] The cross product of vectors $\vec{u}$ and $\vec{v}$ in $\Re^3$, denoted by $cross(\vec{u}, \vec{v}) = \vec{u} \times \vec{v}$, is the vector that satisfies the following geometric properties.

1. the direction of $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$
2. the magnitude of $\vec{u} \times \vec{v}$ is $\|\vec{u}\|\|\vec{v}\| \sin\theta$, where θ is the angle between the vectors $\vec{u}$ and $\vec{v}$
3. if you point the fingers of your *right* hand in the direction of $\vec{u}$ and curl them in the direction of $\vec{v}$, then your thumb points in the direction of $\vec{u} \times \vec{v}$.

The last condition, known as the *right-hand rule*, is required because there are two choices for the direction perpendicular to two given vector; the convention is to chose $\vec{u} \times \vec{v}$ such that condition (3) holds. (See Figure 1.)

Figure 1: Cross product in $\Re^3$



Several important geometric properties of the cross product follow directly from the definition.

Theorem 1 [Geometric properties of cross product in $\Re^3$] Let $\vec{u}$ and $\vec{v}$ be vectors in $\Re^3$. The following properties hold.

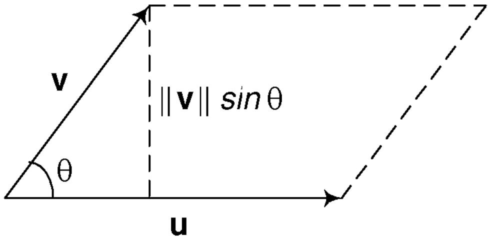
1. $||\vec{u} \times \vec{v}||$ is the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$
2. $\vec{u} \times \vec{v} = 0$ if and only if $\vec{u}$ is parallel to $\vec{v}$

Proof:

1. From Figure 2, $Area = (base)(height) = ||\vec{u}||(||\vec{v}||sin\theta) = ||\vec{u} \times \vec{v}||$
2. $\vec{u} \parallel \vec{v} \Longleftrightarrow \theta = 0$ or $\theta = \pi \Longleftrightarrow \sin\theta = 0 \Longleftrightarrow \vec{u} \times \vec{v} = 0$

∎

Figure 2: Area of parallelogram in $\Re^3$



We can also derive some algebraic properties of the cross product which will be useful later.

Theorem 2 [Algebraic properties of cross product in $\Re^3$] Let $\vec{u}$ and $\vec{v}$ be vectors in $\Re^3$, and let c be a real number. The following properties hold.

1. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$
2. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

Proof:

1. $c(\vec{u} \times \vec{v})$ is perpendicular to $c\vec{u}$ and $\vec{v}$, its length is $||c(\vec{u} \times \vec{v})|| = c||\vec{u}||||\vec{v}|| \sin\theta = ||c\vec{u}||||\vec{v}|| \sin\theta = ||(c\vec{u}) \times \vec{v}||$, and it obeys the *right-hand rule.*
2. Apply the *right-hand rule* to $\vec{v} \times \vec{u}$: reversing the order of the product reverses the direction the hand rotates, and hence the direction the thumb points, which is the direction of the cross product; hence, the direction of the cross product is reversed.

∎

Note that by combining (1) and (2) above, we also see that $\vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$. We now prove one of the most fundamental properties of the cross product — the distributive law. This property will allow us to derive a completely algebraic description of the cross product.

Theorem 3 [Distributive property of cross product in $\Re^3$] Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\Re^3$. The following properties hold.

1. $\vec{w} \times (\vec{u} + \vec{v}) = \vec{w} \times \vec{u} + \vec{w} \times \vec{v}$
2. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

Proof: (1) implies (2) since, by the previous theorem,

$$(\vec{u} + \vec{v}) \times \vec{w} = -\vec{w} \times (\vec{u} + \vec{v}) = -\vec{w} \times \vec{u} + (-\vec{w} \times \vec{v}) = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}.$$

Thus we only need to prove (1): We need to show that $(\vec{w} \times \vec{u} + \vec{w} \times \vec{v})$ is perpendicular to $\vec{w}$ and $\vec{u} + \vec{v}$, that it gives the area of the parallelogram spanned by $\{\vec{w}, \vec{u} + \vec{v}\}$, and that it satisfied the *right-hand rule*. To prove that it satisfies the first condition, orthogonality, observe that,

$$\vec{w} \cdot (\vec{w} \times \vec{u} + \vec{w} \times \vec{v}) = \vec{w} \cdot (\vec{w} \times \vec{u}) + \vec{w} \cdot (\vec{w} \times \vec{v}) = 0 + 0 = 0$$

and that,

$$\begin{aligned}
(\vec{u} + \vec{v}) \cdot (\vec{w} \times \vec{u} + \vec{w} \times \vec{v}) &= (\vec{u} + \vec{v}) \cdot (\vec{w} \times \vec{u}) + (\vec{u} + \vec{v}) \cdot (\vec{w} \times \vec{v}) \\
&= \vec{u} \cdot (\vec{w} \times \vec{u}) + \vec{v} \cdot (\vec{w} \times \vec{u}) + \vec{u} \cdot (\vec{w} \times \vec{v}) + \vec{v} \cdot (\vec{w} \times \vec{v}) \\
&= 0 + \vec{v} \cdot (\vec{w} \times \vec{u}) + \vec{u} \cdot (\vec{w} \times \vec{v}) + 0 \\
&= ||\vec{v}||||\vec{w} \times \vec{u}|| \cos \alpha + ||\vec{u}||||\vec{w} \times \vec{v}|| \cos \beta \\
&= ||\vec{v}||(||\vec{w}||||\vec{u}|| \sin \alpha') \cos \alpha + ||\vec{u}||(||\vec{w}||||\vec{v}|| \sin \beta') \cos \beta \\
&= ||\vec{u}||||\vec{v}||||\vec{w}||(\sin \alpha' \cos \alpha + \sin \beta' \cos \beta)
\end{aligned}$$

where

α is the angle between $\vec{v}$ and $\vec{w} \times \vec{u}$
α' is the angle between $\vec{u}$ and $\vec{w}$
β is the angle between $\vec{u}$ and $\vec{w} \times \vec{v}$
β' is the angle between $\vec{v}$ and $\vec{w}$.

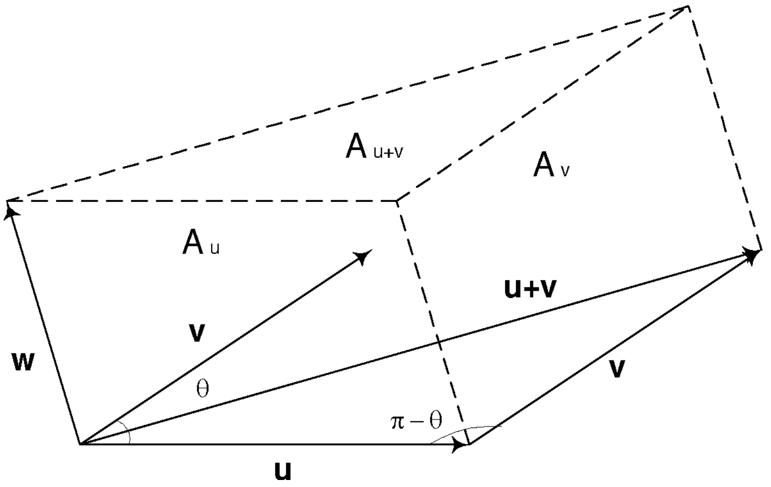
Considering the geometry of the situation, we see that $\sin \alpha' \cos \alpha + \sin \beta' \cos \beta = 0$ for all angles $\alpha, \alpha', \beta, \beta'$ satisfying the above conditions. Thus, $(\vec{u} + \vec{v}) \cdot (\vec{w} \times \vec{u} + \vec{w} \times \vec{v}) = 0$, and the first condition is satisfied.

Next, to prove that the magnitude of $(\vec{w} \times \vec{u} + \vec{w} \times \vec{v})$ is the area of the parallelogram spanned by $\{\vec{w}, \vec{u} + \vec{v}\}$, observe that

$$\begin{aligned}
\|\vec{w} \times \vec{u} + \vec{w} \times \vec{v}\|^2 &= \|\vec{w} \times \vec{u}\|^2 + \|\vec{w} \times \vec{v}\|^2 + 2(\vec{w} \times \vec{u}) \cdot (\vec{w} \times \vec{v}) \\
&= \|\vec{w} \times \vec{u}\|^2 + \|\vec{w} \times \vec{v}\|^2 + 2\|\vec{w} \times \vec{u}\|\|\vec{w} \times \vec{v}\| \cos\theta \\
&= A_u^2 + A_v^2 - 2A_u A_v \cos(\pi - \theta) \\
&= (h\|\vec{u}\|)^2 + (h\|\vec{v}\|)^2 - 2(h\|\vec{u}\|)(h\|\vec{v}\|) \cos(\pi - \theta) \\
&= h^2(\|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos(\pi - \theta)) \\
&= h^2\|\vec{u} + \vec{v}\|^2 \\
&= A_{u+v}^2 \\
&= \|\vec{w} \times (\vec{u} + \vec{v})\|^2.
\end{aligned}$$

Here, θ is the angle between $\vec{u}$ and $\vec{v}$, which is equal to the angle between $\vec{w} \times \vec{u}$ and $\vec{w} \times \vec{v}$, A_u is the area of the parallelogram spanned by $\{\vec{w}, \vec{u}\}$, A_v is the area of the parallelogram spanned by $\{\vec{w}, \vec{v}\}$, A_{u+v} is the area of the parallelogram spanned by $\{\vec{w}, \vec{u} + \vec{v}\}$, and h is the common "height" of all three parallelograms. We have applied the *Law of Cosines* twice, first to expand the initial term, and then to re-group the resulting expression. Refer to Figure 3 for further insight.

Figure 3: Distributive property of cross product in $\Re^3$



Finally, to establish that the cross product satisfies the *right-hand rule*, we only need to observe that we have not changed the order of the product in applying the distributive law, and so since $\vec{w} \times \vec{u}$ and $\vec{w} \times \vec{v}$ satisfy the *right-hand rule*, so must $(\vec{w} \times \vec{u} + \vec{w} \times \vec{v})$. ■

As already stated, the definition we gave at the beginning of this section provides a geometric description of the cross product. Now we would like to derive a completely algebraic description; i.e., given two vectors written in component form relative to the standard basis, we would like to compute the components of the cross product. This will allow us to easily compute cross products of any two vectors expressed in standard form. We tackle this in the following theorem.

Remember that the orthogonal unit vectors $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ point in the directions of the three x-, y-, and z- coordinate axes. These three vectors form an *orthonormal* basis for $\Re^3$, so that every vector $\vec{v} \in \Re^3$ can be written as a *unique* linear combination of the basis vectors $\{\vec{\imath}, \vec{\jmath}, \vec{k}\}$: $\vec{v} = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}$.

Theorem 4 [Determinant expression for cross product in $\Re^3$] Let $\vec{u} = u_1\vec{\imath} + u_2\vec{\jmath} + u_3\vec{k}$ and $\vec{v} = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}$ be vectors in $\Re^3$. Then the cross product of $\vec{u}$ and $\vec{v}$ is obtained by formally evaluating the determinant

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \vec{\imath} & \vec{\jmath} & \vec{k} \end{vmatrix} \quad or \quad \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Proof: The second expression (with the unit vectors in the first row) is more common than the first (with the unit vectors in the last row) in most treatments of the cross product in $\Re^3$. We can easily see that they are equivalent by noting that one can be transformed into the other by *two* elementary row swaps, each of which negates the expression. And thus, since $(-1)(-1) = 1$, they are equivalent. We will see later that, in general, these two forms are *not* equivalent.

To derive the determinant expression for the cross product, we start with the following observations about the unit vectors $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ that form the orthonormal basis for $\Re^3$:

$$\begin{array}{lll} \vec{\imath} \times \vec{\imath} = 0 & \vec{\jmath} \times \vec{\jmath} = 0 & \vec{k} \times \vec{k} = 0 \\ \vec{\imath} \times \vec{\jmath} = \vec{k} & \vec{\imath} \times \vec{k} = -\vec{\jmath} & \vec{k} \times \vec{\jmath} = -\vec{\imath} \\ \vec{\jmath} \times \vec{\imath} = -\vec{k} & \vec{k} \times \vec{\imath} = \vec{\jmath} & \vec{\jmath} \times \vec{k} = \vec{\imath} \end{array}$$

(Note that the cross product of a vector with itself is the zero vector.) This multiplication table can be remembered by considering the repeating sequence of unit vectors

$$\underbrace{\vec{\imath},\ \vec{\jmath},\ \vec{k}}_{\vec{\imath}\times\vec{\jmath}=\vec{k}},\ \vec{\imath},\ \vec{\jmath},\ \underbrace{\vec{k},\ \vec{\imath},\ \vec{\jmath}}_{\vec{k}\times\vec{\imath}=\vec{\jmath}},\ \vec{k},\ \ldots$$

and noting that the product of any two consecutive vectors in the list is the next vector in the list. The rest of the products follow from considering the anti-commutative nature of the cross product, Theorem 2-2.

Now we write vectors $\vec{u}$ and $\vec{v}$ as linear combinations of the basis vectors $\{\vec{\imath}, \vec{\jmath}, \vec{k}\}$ and expand the cross product expression by repeatedly applying the distributive property, Theorem 3. And then using the above identities to simplify the resulting expression.

$$\begin{aligned}
\vec{u} \times \vec{v} &= (u_1\vec{\imath} + u_2\vec{\jmath} + u_3\vec{k}) \times (v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}) \\
&= (u_1\vec{\imath}) \times (v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}) + (u_2\vec{\jmath}) \times (v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}) + (u_3\vec{k}) \times (v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}) \\
&= u_1v_1(\vec{\imath} \times \vec{\imath}) + u_1v_2(\vec{\imath} \times \vec{\jmath}) + u_1v_3(\vec{\imath} \times \vec{k}) \\
&\quad + u_2v_1(\vec{\jmath} \times \vec{\imath}) + u_2v_2(\vec{\jmath} \times \vec{\jmath}) + u_2v_3(\vec{\jmath} \times \vec{k}) \\
&\quad + u_3v_1(\vec{k} \times \vec{\imath}) + u_3v_2(\vec{k} \times \vec{\jmath}) + u_3v_3(\vec{k} \times \vec{k}) \\
&= (u_1v_2)\vec{k} - (u_1v_3)\vec{\jmath} - (u_2v_1)\vec{k} + (u_2v_3)\vec{\imath} + (u_3v_1)\vec{\jmath} - (u_3v_2)\vec{\imath} \\
&= (u_2v_3 - u_3v_2)\vec{\imath} - (u_1v_3 - u_3v_1)\vec{\jmath} + (u_1v_2 - u_2v_1)\vec{k} \\
&= \vec{\imath}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}.
\end{aligned}$$

It is important to observe that this computation can be carried out because the distributive property holds for cross products in $\Re^3$. Notice that the last expression is just the determinant that appears in this theorem, evaluated by co-factor expansion along the first row. ∎

3 Going Down – Cross Product in 2-Space

To extend the notion of a cross product to higher (and lower) dimensions, consider the defining characteristic of a cross product — an essential feature without which it would not have the properties to reasonably call it a cross product: orthogonality. Orthogonality is the *defining* property of the cross product. The cross product of two vectors in $\Re^3$ is orthogonal (perpendicular) to the two input vectors. It would seem natural that the cross product in two dimensions, defined over $\Re^2$, would result in a vector perpendicular to the inputs. But some thought quickly convinces one that there is *no* vector that is perpendicular to two non-parallel vectors in $\Re^2$ — there just isn't enough "room." But given a *single* vector in $\Re^2$, there are only *two* possible directions for the vector orthogonal to it: namely any vector on the line perpendicular to the given vector. If we then restrict the length of the resultant vector to be the same as the length of the given vector, only two possible vectors remain: the one obtained from the given vector by a *counterclockwise* rotation about their common point, and the vector opposite that one, obtained from the given vector by a *clockwise* rotation. Just as in the 3-dimensional case, we have two possible choices for the cross product. We must make an arbitrary choice, just like we did in $\Re^3$ with the right hand rule, to chose one of them in order to define the cross product uniquely. Since the convention in mathematics is for angles to be measured counterclockwise, let us chose the vector that is obtained from the given vector by a *counterclockwise* rotation. This thinking is formalized in the following definition of the cross product in $\Re^2$.

Definition 2 [Cross product in $\Re^2$] Let $\vec{u}$ be a vector in $\Re^2$. Then the *cross of* $\vec{u}$, denoted by *cross*$(\vec{u})$ or $\chi(\vec{u})$, is the vector obtained from $\vec{u}$ by a counterclockwise rotation of $\pi/2$ radians and with the same magnitude as $\vec{u}$ (i.e., $\|\chi(\vec{u})\| = \|\vec{u}\|$).

To obtain an algebraic expression for $\chi(\vec{u})$, consider Figure 4.

Theorem 5 [Determinant expression for cross product in $\Re^2$] Let $\vec{u} = (x, y) = u_1\vec{\imath} + u_2\vec{\jmath}$ be a vector in $\Re^2$, then

$$\chi(\vec{u}) = (-y, x) = -u_2\vec{\imath} + u_1\vec{\jmath} = \begin{vmatrix} u_1 & u_2 \\ \vec{\imath} & \vec{\jmath} \end{vmatrix}.$$

Proof: From Figure 4, since $\theta' = \theta + \pi/2$,

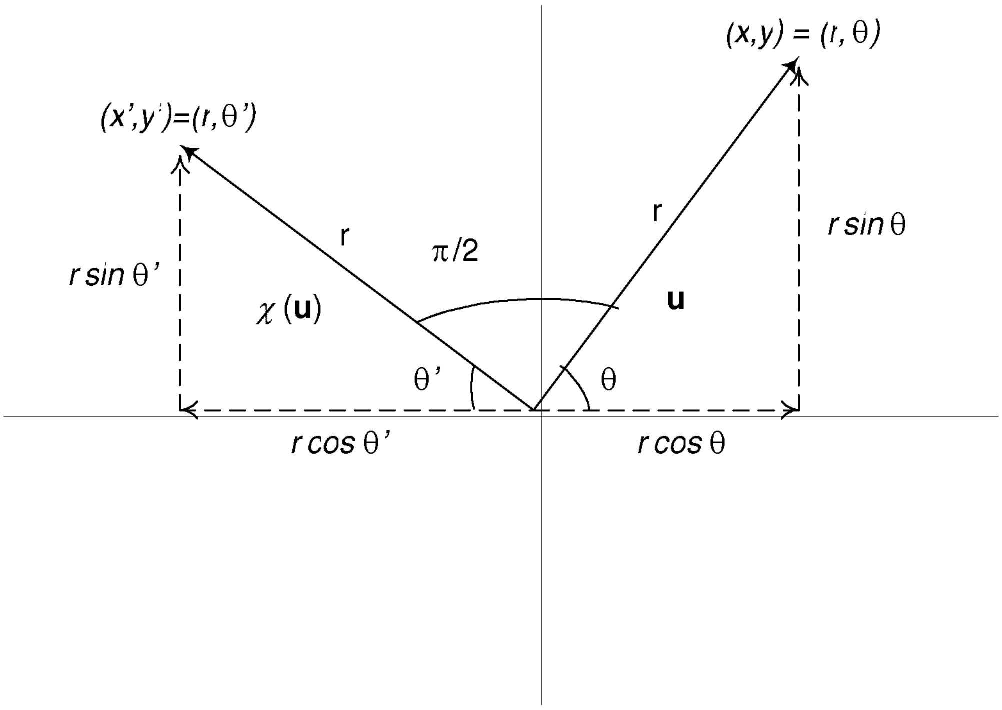
$$\begin{aligned} x' &= r\cos\theta' = r\cos(\theta + \pi/2) = r(\cos\theta\cos\pi/2 - \sin\theta\sin\pi/2) = -r\sin\theta = -y \\ y' &= r\sin\theta' = r\sin(\theta + \pi/2) = r(\sin\theta\cos\pi/2 + \cos\theta\sin\pi/2) = r\cos\theta = x \end{aligned}$$

and we see that $\chi(x, y) = (x', y') = (-y, x)$ as required.

Alternatively, recall the matrix for a rotation of coordinates,

$$P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Figure 4: Cross product in $\Re^2$



that transforms a vector relative to the standard basis to a vector relative to a basis rotated θ radians counterclockwise. This rotation matrix also gives the coordinates of a vector rotated by θ radians. With $\theta = \pi/2$, the rotation matrix becomes,

$$P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Pre-multiplying the vector $\vec{u}^T$ by P gives the vector $\chi(\vec{u})^T$ as desired:

$$P\,\vec{u}^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} = \chi(\vec{u})^T.$$

∎

We can easily verify that the formula given in Theorem 5 does indeed satisfy the properties we claimed:

1. $||\chi(\vec{u})|| = \sqrt{(-u_2)^2 + (u_1)^2} = \sqrt{u_1^2 + u_2^2} = ||\vec{u}||$
2. $\vec{u} \cdot \chi(\vec{u}) = (u_1\vec{\imath} + u_2\vec{\jmath}) \cdot (-u_2\vec{\imath} + u_1\vec{\jmath}) = -u_1u_2 + u_2u_1 = 0 \Rightarrow \vec{u} \perp \chi(\vec{u})$

Now that we are in possession of this explicit expression for the cross product in $\Re^2$, we may realize that many mathematical situations in $\Re^2$ that we did not normally associate with the cross product, are closely related to it.

4 Going Up – Cross Product in 4-Space

Now we turn our consideration to 4-space. We are faced with a similar problem to the one in $\Re^2$ — the cross product of two vectors is not well defined. Given two vectors in $\Re^4$, there is a whole *plane* of vectors perpendicular to them. This is because $\Re^4$ has more "room" than $\Re^3$. Some thought leads to the conclusion that a minimum of *three* linearly independent vectors are required to define the cross product direction unambiguously. We are still left with the problem of choosing a particular orientation, and defining the magnitude of the cross product. Choosing the orientation is arbitrary, as it is in $\Re^2$ and $\Re^3$: one direction is arbitrarily labeled "positive" or "right" and the other "negative" or "left." We can not picture this using a system like the *right-hand rule* as we did in $\Re^3$ because it is impossible to visualize or to handle four dimensions directly. Finally, a natural definition of the magnitude of the cross product is that it gives the 3-volume of the 4-dimensional parallelepiped spanned by the three input vectors.

Definition 3 [Cross product in $\Re^4$] Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\Re^4$. Then the *cross product* of $\vec{u}$, $\vec{v}$, and $\vec{w}$, denoted by

$$cross(\vec{u}, \vec{v}, \vec{w}) = \chi(\vec{u}, \vec{v}, \vec{w}),$$

is a vector in $\Re^4$ such that

1. $\chi(\vec{u}, \vec{v}, \vec{w}) \perp$ span$(\vec{u}, \vec{v}, \vec{w})$: $\chi(\vec{u}, \vec{v}, \vec{w}) \cdot \vec{u} = 0$, $\chi(\vec{u}, \vec{v}, \vec{w}) \cdot \vec{v} = 0$, and $\chi(\vec{u}, \vec{v}, \vec{w}) \cdot \vec{w} = 0$
2. $\|\chi(\vec{u}, \vec{v}, \vec{w})\|$ is the 3-volume of the 4-dimensional parallelepiped spanned by $\{\vec{u}, \vec{v}, \vec{w}\}$
3. if, when you curl the fingers of your *right* hand from $\vec{u}$ into $\vec{v}$, your thumb points in the direction of $\vec{w}$, then $\chi(\vec{u}, \vec{v}, \vec{w})$ points in the 'positive' or 'right' direction; otherwise it points in the 'negative' or 'left' direction

Condition (3) is the analogue of the *right-hand rule* in $\Re^3$. Throughout this section, θ will designate the angle between the vectors $\vec{u}$ and $\vec{v}$, and ϕ will designate the angle between $\vec{w}$ and the plane determined by $\vec{u}$ and $\vec{v}$. (Refer to Figure 5.)

Several geometric properties analogous to the ones we are familiar with in $\Re^3$ follow immediately.

Theorem 6 [Geometric properties of cross product in $\Re^4$] Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\Re^4$. The following properties hold.

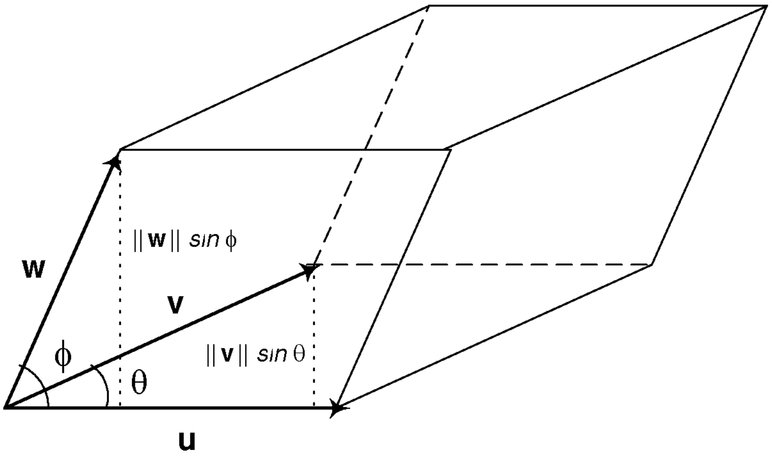
1. $\|\chi(\vec{u},\vec{v},\vec{w})\| = \|\vec{u}\|\|\vec{v}\|\|\vec{w}\| \sin\theta \sin\phi$, where θ and ϕ are as defined above
2. $\chi(\vec{u},\vec{v},\vec{w}) = \vec{0}$ if and only if the set $\{\vec{u},\vec{v},\vec{w}\}$ is linearly dependent

Proof:

1. From Figure 5, $\|\chi(\vec{u},\vec{v},\vec{w})\| = Volume(\vec{u},\vec{v},\vec{w}) = (Base)(height) = (\|\vec{u}\|\|\vec{v}\| sin\theta)(\|\vec{w}\| \sin\phi) = \|\vec{u}\|\|\vec{v}\|\|\vec{w}\| \sin\theta \sin\phi$.
2. If $\{\vec{u},\vec{v},\vec{w}\}$ is linearly dependent, the volume of the parallelepiped spanned by $\{\vec{u},\vec{v},\vec{w}\}$ is 0, which implies that $\|\chi(\vec{u},\vec{v},\vec{w})\| = 0$. Conversely, if $\chi(\vec{u},\vec{v},\vec{w}) = 0$, then the volume of the parallelepiped spanned by $\{\vec{u},\vec{v},\vec{w}\}$ is 0, and hence either $\theta = 0$ or $\phi = 0$, and the set must be linearly dependent.

∎

Figure 5: Volume of parallelepiped in $\Re^4$



Several important algebraic properties are also apparent.

Theorem 7 [Algebraic properties of cross product in $\Re^4$] Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\Re^4$, and let a, b, and c be real numbers. The following properties hold.

1. $\chi(a\vec{u}, b\vec{v}, c\vec{w}) = (abc)\chi(\vec{u}, \vec{v}, \vec{w})$
2. $\chi(\vec{u}, \vec{v}, \vec{w}) = -\chi(\vec{v}, \vec{u}, \vec{w})$
3. $\chi(\vec{u}, \vec{v}, \vec{w}) = -\chi(\vec{u}, \vec{w}, \vec{v})$
4. $\chi(\vec{u}, \vec{v}, \vec{w}) = -\chi(\vec{w}, \vec{v}, \vec{u})$

Proof:

1. $(abc)\chi(\vec{u}, \vec{v}, \vec{w})$ is perpendicular to $a\vec{u}$, $b\vec{v}$, and $c\vec{w}$, $\|(abc)\chi(\vec{u}, \vec{v}, \vec{w})\|$ is the 3-volume of the parallelepiped spanned by $\{a\vec{u}, b\vec{v}, c\vec{w}\}$, and it satisfies the *right-hand rule*
2. (2)-(4) follow from condition (3) in the definition of the cross product: we see that swapping any two vectors in the product reverses the direction of $\chi(\vec{u}, \vec{v}, \vec{w})$ from 'positive' to 'negative.'

■

Now we prove the distributive law for cross products in $\Re^4$. This property is of fundamental importance, and will be used later in our derivation of the algebraic description of the cross product. We need to write three distribute laws instead of just one because the cross product is not commutative.

Theorem 8 [Distributive property of cross product in $\Re^4$] Let $\vec{u}$, $\vec{v}$, $\vec{w}$, and $\vec{z}$ be vectors in $\Re^4$. The following properties hold.

1. $\chi(\vec{u} + \vec{z}, \vec{v}, \vec{w}) = \chi(\vec{u}, \vec{v}, \vec{w}) + \chi(\vec{z}, \vec{v}, \vec{w})$
2. $\chi(\vec{u}, \vec{v} + \vec{z}, \vec{w}) = \chi(\vec{u}, \vec{v}, \vec{w}) + \chi(\vec{u}, \vec{z}, \vec{w})$
3. $\chi(\vec{u}, \vec{v}, \vec{w} + \vec{z}) = \chi(\vec{u}, \vec{v}, \vec{w}) + \chi(\vec{u}, \vec{v}, \vec{z})$

Proof: (1) implies (2) since, by Theorem 7-2,

$$\chi(\vec{u}, \vec{v} + \vec{z}, \vec{w}) = -\chi(\vec{v} + \vec{z}, \vec{u}, \vec{w}) = -\chi(\vec{v}, \vec{u}, \vec{w}) + (-\chi(\vec{z}, \vec{u}, \vec{w})) = \chi(\vec{u}, \vec{v}, \vec{w}) + \chi(\vec{u}, \vec{z}, \vec{w}).$$

And similarly, (1) implies (3) since, by Theorem 7-4,

$$\chi(\vec{u}, \vec{v}, \vec{w} + \vec{z}) = -\chi(\vec{w} + \vec{z}, \vec{v}, \vec{u}) = -\chi(\vec{w}, \vec{v}, \vec{u}) + (-\chi(\vec{z}, \vec{v}, \vec{u})) = \chi(\vec{u}, \vec{v}, \vec{w}) + \chi(\vec{u}, \vec{v}, \vec{z}).$$

Thus, we only need to prove (1): we will take an indirect approach here. First, by inspection, observe that χ distributes for all permutations of the unit vectors $\{\vec{\imath}, \vec{\jmath}, \vec{k}, \vec{l}\}$.

We will use this fact in the derivation of an explicit expression for the cross product. The most general case will then follow immediately as a corollary. $\blacksquare$

We now follow the same thinking as in $\Re^2$ and $\Re^3$: we want to derive an expression for the coordinates of $\chi(\vec{u},\vec{v},\vec{w})$ in terms of the coordinates of $\vec{u}$, $\vec{v}$, and $\vec{w}$, relative to the standard basis. We proceed very much as we did in $\Re^3$. First, we must introduce a new unit vector $\vec{l}$ that points in the "4th dimension," i.e., a vector of unit length perpendicular to the unit vectors $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ in $\Re^4$. Again, there are two such unit vectors, and we are at liberty to chose one direction and arbitrarily label it as "positive." Being more precise, let

$$\vec{l} = \chi(\vec{\imath},\vec{\jmath},\vec{k}).$$

We see that $\{\vec{\imath},\vec{\jmath},\vec{k},\vec{l}\}$ forms an *orthonormal* basis for $\Re^4$; i.e., any vector $\vec{v} \in \Re^4$ can be written as a *unique* linear combination of the basis vectors: $\vec{v} = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k} + v_4\vec{l}$. We thus see that

$$\begin{array}{lll}
\chi(\vec{\imath},\vec{\jmath},\vec{k}) = \vec{l} & \chi(\vec{\jmath},\vec{\imath},\vec{k}) = -\vec{l} & \chi(\vec{k},\vec{\jmath},\vec{\imath}) = -\vec{l} \\
\chi(\vec{\imath},\vec{k},\vec{\jmath}) = -\vec{l} & \chi(\vec{\jmath},\vec{k},\vec{\imath}) = \vec{l} & \chi(\vec{k},\vec{\imath},\vec{\jmath}) = \vec{l}
\end{array}$$

$$\vdots\ (*)$$

The same pattern repeats with all ${}_4P_3 = 24$ permutations of three of the four unit vectors, by Theorems 7-2, 7-3, and 7-4. Observe that the cross product of three of the same vectors is zero; more generally, the cross product of any three vectors where any two of them are identical is zero by Theorem 6-2. For example,

$$\begin{aligned}
\chi(\vec{\imath},\vec{\imath},\vec{\imath}) &= 0 \\
\chi(\vec{\imath},\vec{\imath},\vec{\jmath}) &= 0 \\
\chi(\vec{\imath},\vec{\jmath},\vec{\imath}) &= 0 \\
&\vdots\ (**)
\end{aligned}$$

There are a total of $4^3 = 64$ possible arrangements of the unit vectors in $\Re^4$. ${}_4P_3 = 24$ of them are of the form $(*)$, and the remaining, or $4^3 - {}_4P_3 = 40$, are of the form $(**)$. We can remember this multiplication table by considering the sequence of unit vectors,

$$\underbrace{\vec{\imath},\ \vec{\jmath},\ \vec{k},\ \vec{l},}_{\chi(\vec{\imath},\vec{\jmath},\vec{k})=\vec{l}}\ \vec{\imath},\ \underbrace{\vec{\jmath},\ \vec{k},\ \vec{l},\ \vec{\imath},}_{\chi(\vec{\jmath},\vec{k},\vec{l})=\vec{\imath}}\ \vec{\jmath},\ \underbrace{\vec{k},\ \vec{l},\ \vec{\imath},\ \vec{\jmath},}_{\chi(\vec{k},\vec{l},\vec{\imath})=\vec{\jmath}}\ \vec{k},\ \vec{l},\ \ldots$$

and noting that the product of any three consecutive vectors in this sequence is the next vector in the sequence. The other products follow from considering the anti-commutative nature of the product, Theorem 7.

Given the cross product of all possible permutations of the unit vectors, we can carry out a computation for the cross product of *any* three vectors in $\Re^4$, much as we did in $\Re^3$. We can do this by expanding the product by successively applying the distributive property, Theorem 8 to

$$\vec{z} = \chi(\vec{u},\vec{v},\vec{w})$$

where

$$\begin{aligned}\vec{z} &= z_1\vec{\imath} + z_2\vec{\jmath} + z_3\vec{k} + z_4\vec{l}\\ \vec{u} &= u_1\vec{\imath} + u_2\vec{\jmath} + u_3\vec{k} + u_4\vec{l}\\ \vec{v} &= v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k} + v_4\vec{l}\\ \vec{w} &= w_1\vec{\imath} + w_2\vec{\jmath} + w_3\vec{k} + w_4\vec{l}.\end{aligned}$$

First, we note that there are a total of $4^3 = 64$ terms in this expansion. But the majority, or $4^3 - {}_4P_3 = 40$, are of the form $(**)$, and hence equal to zero. That leaves only ${}_4P_3 = 24$ non-zero terms in the expansion. By carrying out the expansion in an orderly fashion, i.e., by expanding along each component of each of the three vectors in order, we can use the symmetry of the computation to reduce our work. For each component of $\vec{u}$, there are exactly six terms, half of which are positive and half of which are negative, in the expansion. These two sets of three terms correspond to the positive and negative values of the cross product of three of the four unit vectors $\{\vec{\imath}, \vec{\jmath}, \vec{k}, \vec{l}\}$. There are four such sets, one for each component of $\vec{u}$, accounting for the required $6 \cdot 4 = 24$ total non-zero terms. It is a rather routine computation, and only the terms in the $\vec{l}$ direction (the fourth component of the cross product) are shown here.

$$z_4 = u_1v_2w_3 - u_1v_3w_2 - u_2v_1w_3 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1.$$

Note that no fourth component of any of the input vectors appears in this expression; i.e., there are no u_4, v_4, or w_4 terms in the expression for z_4. Note that each of the six terms in the expression for z_4 correspond to each of the six possible ways there are to permute the three vectors $\{\vec{\imath}, \vec{\jmath}, \vec{k}\}$ (excluding $\vec{l}$), with minus signs attached as required by the anti-commutative property of the cross product, Theorem 7.

Now consider the symbolic determinant given by

$$\begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \\ \vec{\imath} & \vec{\jmath} & \vec{k} & \vec{l} \end{vmatrix}.$$

Calculating the cofactor of the lower-left entry $\vec{l}$, we see that they correspond to the fourth component of the cross product of $\vec{u}$, $\vec{v}$, and $\vec{w}$, as calculated above.

$$\begin{aligned} Cofactor(\vec{l}) &= (-1)^{4+4}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= u_1\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \\ &= u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1) \\ &= u_1v_2w_3 - u_1v_3w_2 - u_2v_1w_3 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1 \\ &= z_4. \end{aligned}$$

A similar computation for the other three components of $\vec{z}$ confirms that the determinant given above does in fact give the components of the cross product. There is a remarkable amount of symmetry in this computation, and different terms group together to give the determinant expression given above. We see immediately that the result is directly analogous to the result in $\Re^3$. This strong similarity between the cross product expressions in $\Re^2$, $\Re^3$, and $\Re^4$ is indeed fascinating, and suggests that there is a deeper connection between determinants and cross products, as we shall see in the next section. The following theorem states and summarizes the results for the cross product in $\Re^4$.

Theorem 9 [Determinant expression for cross product in $\Re^4$] Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\Re^4$, with components (u_i), (v_i), and (w_i) respectively. Then the cross product of $\vec{u}$, $\vec{v}$, and $\vec{w}$, is

$$\chi(\vec{u}, \vec{v}, \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \\ \vec{\imath} & \vec{\jmath} & \vec{k} & \vec{l} \end{vmatrix}.$$

[Note that the unit vectors appear as the last row in this determinant. Also note that *three* elementary row swaps are required to place the unit vectors in the first row (and to shift the other three rows down), so these expressions are *not* equivalent, as they are in $\Re^3$. Instead, they are negations of each other, since $(-1)^3 = -1$.]

[Also note that Theorem 8-1 now follows in its full generality as promised because the determinant function is linear in each row-vector.]

5 Up, Up, and Away! – Cross Product in N-Space

The central role that the determinant plays in the last three examples of the cross product in two, three, and four dimensions suggests that an analogous $n \times n$ determinant gives the cross product in $\Re^n$. We can test out this conjecture by simply evaluating the (n-dimensional) dot product between each of the $n-1$ vectors and their cross product, and checking that it evaluates to zero. This provides a purely algebraic method of verifying the conjecture, and allows us to bypass the problem of trying to visualize n-dimensional space. This thinking is formalized in the following definition, and leads to the next theorem.

Definition 4 [Cross product in $\Re^n$] Let $V = \Re^n$ be the n-dimensional Euclidean space with standard dot product, let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1}$ be $n-1$ vectors in V, and let c be a real number. Let

$$\chi : V^{n-1} \mapsto V$$

be a function. We say that χ is a *cross product* if the following conditions are satisfied.

1. $\chi(\vec{u}_1, \ldots, \vec{u}_i, \ldots, \vec{u}_{n-1}) \cdot \vec{u}_i = 0$ for $i = 1, 2, \ldots, n-1$
2. $\|\chi(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1})\| = Volume(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1})$
3. $\chi(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{n-1}) = \vec{e}_n$ where $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is an orthonormal basis for V

The first condition says that the cross product is orthogonal to each input vector. The second condition describes the length of the cross product as the $(n-1)$-volume of the n-dimensional parallelepiped spanned by the input vectors. The last condition establishes a positive orientation for the basis vectors, and hence for the cross product, much like the *right-hand rule* does in $\Re^3$. These three properties of the cross product — direction, length, and orientation — define a unique cross product for each set of input vectors.

Theorem 10 [Determinant expression for cross product in $\Re^n$] Let $V = \Re^n$ be the n-dimensional Euclidean space with standard orthonormal basis $B = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$, and let

$$\begin{aligned}
\vec{u}_1 &= u_{11}\vec{e}_1 + u_{12}\vec{e}_2 + \cdots + u_{1n}\vec{e}_n \\
\vec{u}_2 &= u_{21}\vec{e}_1 + u_{22}\vec{e}_2 + \cdots + u_{2n}\vec{e}_n \\
&\vdots \\
\vec{u}_{n-1} &= u_{n-1,1}\,\vec{e}_1 + u_{n-1,2}\,\vec{e}_2 + \cdots + u_{n-1,n}\,\vec{e}_n
\end{aligned}$$

be $n-1$ vectors in V.

The cross product of $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1}\}$ is given by

$$\chi(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1}) = \begin{vmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1,1} & u_{n-1,2} & \cdots & u_{n-1,n} \\ \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{vmatrix}.$$

Proof: Let $\vec{z} = z_1\vec{e_1} + z_2\vec{e_2} + \cdots + z_n\vec{e_n} = \chi(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1})$. We want to show that $\vec{z}$ satisfies the three properties given in the definition of the cross product — direction, length, and orientation. To prove the orthogonality of the cross product, we need to show that $\vec{u}_i \cdot \vec{z} = 0$ for all $\vec{u}_i$. Observe that

$$\begin{aligned}
\vec{u}_i \cdot \vec{z} &= (u_{i1}\vec{e_1} + u_{i2}\vec{e_2} + \cdots + u_{in}\vec{e_n}) \cdot (z_1\vec{e_1} + z_2\vec{e_2} + \cdots + z_n\vec{e_n}) \\
&= (u_{i1}\vec{e_1} + u_{i2}\vec{e_2} + \cdots + u_{in}\vec{e_n}) \cdot \begin{vmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1,1} & u_{n-1,2} & \cdots & u_{n-1,n} \\ \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{vmatrix} \\
&= \begin{vmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1,1} & u_{n-1,2} & \cdots & u_{n-1,n} \\ u_{i1} & u_{i2} & \cdots & u_{in} \end{vmatrix}.
\end{aligned}$$

We see that this determinant is zero since the last row is the same as row i. (By subtracting row i from the last row, we get a row of zeroes, which we see gives a determinant of zero by evaluating along that row.) This establishes that the cross product expression given above is perpendicular to each input vector.

To establish that its length agrees with our definition, we use a result from linear algebra which states that the $n \times n$ determinant gives the n-volume of the n-dimensional parallelepiped spanned by its n row vectors. (See the Appendix for a proof of this theorem.)

We want to show that the cross product gives the $(n-1)$-volume of the n-dimensional parallelepiped spanned by $\{\vec{u}_i\}$. Chose a vector $\vec{v}$ that is *not* in the span of $\{\vec{u}_i\}$; such a vector is guaranteed to exist because in an n-dimensional vector space, one can always find n linearly independent vectors. Let

$$Vol_n = Volume(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1}, \vec{v})$$

be the n-volume of the n-dimensional parallelepiped spanned by $\{\vec{u}_i, \vec{v}\}$. Observe that

$$Vol_n = |\vec{v} \cdot \chi(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1})| = |\vec{v} \cdot \vec{z}|$$

by a similar computation on determinants as performed above. Then,

$$\begin{aligned}
Vol_n &= \|\vec{v}\|\|\vec{z}\| \cos\theta, \text{ where } \theta \text{ is the angle between } \vec{z} \text{ and } \vec{v} \\
&= (\|\vec{v}\| \cos\theta)\|\vec{z}\|.
\end{aligned}$$

Observe that $\|\vec{v}\| \cos\theta$ is the magnitude of the projection of $\vec{v}$ onto the vector $\vec{z}$, which is orthogonal to span$(\vec{u}_i)$ (as we proved above). Thus,

$$Vol_n = \|Proj_{\,\vec{z}}\,(\vec{v})\|\ \|\vec{z}\| = (height)(Base) = (height)(Vol_{n-1}),$$

which implies that the $(n-1)$-volume of the base is

$$\|\vec{z}\| = Vol_{n-1} = Volume(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1}).$$

∎

This remarkable result in $\Re^n$ confirms our earlier suspicions about the close connection between the cross product and the determinant. Indeed, this proof suggests that they are almost identical. The determinant is the algebraic face of the cross product, and the cross product is the geometric face of the determinant. This is a beautiful connection between a purely algebraic object, the determinant of matrices, and a purely geometric object, the cross product of vectors. This is perhaps the greatest unity in mathematics. Continuing our study of the cross product, we can prove many important algebraic properties of the n-dimensional cross-product.

Theorem 11 [Algebraic properties of cross product in $\Re^n$] Let $\vec{u}_1$, $\vec{u}_2$, …, $\vec{u}_{n-1}$, and $\vec{v}$ be vectors in $\Re^n$, and let c be a real number. The following properties hold.

1. $\chi(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1}) = 0$ if and only if the set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1}\}$ is linearly dependent
2. $\chi(\vec{u}_1, \ldots, c\vec{u}_i, \ldots, \vec{u}_{n-1}) = c\chi(\vec{u}_1, \ldots, \vec{u}_i, \ldots, \vec{u}_{n-1})$ for $i = 1, 2, \ldots, n-1$
3. $\chi(\vec{u}_1, \ldots, \vec{u}_i + \vec{v}, \ldots, \vec{u}_{n-1}) = \chi(\vec{u}_1, \ldots, \vec{u}_i, \ldots, \vec{u}_{n-1}) + \chi(\vec{u}_1, \ldots, \vec{v}, \ldots, \vec{u}_{n-1})$ for $i = 1, 2, \ldots, n-1$
4. $\chi(\vec{u}_1, \ldots, \vec{u}_i, \ldots, \vec{u}_j, \ldots, \vec{u}_{n-1}) = -\chi(\vec{u}_1, \ldots, \vec{u}_j, \ldots, \vec{u}_i, \ldots, \vec{u}_{n-1})$ for $i, j = 1, 2, \ldots, n-1$ and $i \neq j$

Proof: These properties follow directly from properties of determinants. See Larson and Edwards [5, pp. 120–137] for proofs of the various properties of determinants alluded to here and in this section.

1. This property follows from the fact that the determinant is zero whenever the row vectors are linearly dependent. (To see this, simply subtract appropriate scalar multiples of the rows from a particular row to obtain a row of zeroes, from which if follows the determinant is zero.)
2. This property follows from the property of determinants that state that a common factor from one row (or column) can be factored out of the determinant expression.
3. This property is similar to (2); they both follow from the linearity of the determinant in each row vector.
4. This property follows from the property of determinants that swapping any two rows (or columns) negates the value of the determinant. (We have already used this property in conjunction with the question of where to place the unit vectors in the determinant expression for the cross product.)

∎

To complete and round-off our discussion of the n-dimensional cross product, we will prove a generalization of a theorem known in three dimensions as Lagrange's Identity, after the 18th century French mathematician Joseph L. Lagrange. This identity establishes an interesting connection between the cross product and the dot product of vectors.

Theorem 12 [Lagrange's Indentiy in $\Re^n$] Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1}$ be $n-1$ vectors in $\Re^n$. Then

$$\|\chi(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1})\| = \prod_{i=1}^{n-1} \left\| \vec{u}_i - \sum_{j=1}^{i-1} \frac{\vec{u}_i \cdot \vec{w}_j}{\vec{w}_j \cdot \vec{w}_j} \vec{w}_j \right\|$$

where

$$\vec{w}_j = \vec{u}_j - \sum_{k=1}^{j-1} \left(\frac{\vec{u}_j \cdot \vec{w}_k}{\vec{w}_k \cdot \vec{w}_k} \right) \vec{w}_k.$$

Proof: This may seem like an unwieldy and useless extension of the simple and beautiful result of Lagrange in three-dimensions. It is in fact neither: this rather complicated expression gets to the very heart and soul of the cross product, and even more closely unifies the algebraic and the geometric faces of the cross product.

Let us first observe that the $(n-1)$-volume of the n-dimensional parallelepiped spanned by $\{\vec{u}_i\}$ is given by

$$\|\chi(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1})\| = \|\vec{u}_1\| \|\vec{u}_2\| \cdots \|\vec{u}_{n-1}\| \sin(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{n-2}),$$

where θ_i is the angle between the vector $\vec{u}_i$ and the *hyperspace* spanned by $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{i-1}\}$. This geometric expression for the length of the cross product is very useful in two, three, and even four dimensions, but becomes imprecise in higher dimensions. (See Figures 2 and 5 for the three dimensional and four dimensional cases, respectively.) We can derive a more useful expression for the length of the cross product by considering an alternative method of arriving at the volume of the parallelepiped spanned by $\{\vec{u}_i\}$. If we think of "deforming" the parallelepiped into one with perpendicular sides, then its length is easy to compute: simply multiply the lengths of its sides. We can accomplish this by subtracting off proper scalar multiples (projections) of consecutive vectors. (For the details, see the Appendix.) Hence we "deform" a parallelepiped with sides $\vec{u}_i$ into a parallelepiped (with the *same volume*) with all *perpendicular* sides $\vec{w}_i$, where

$$\begin{aligned}
\vec{w}_1 &= \vec{u}_1 \\
\vec{w}_2 &= \vec{u}_2 - \left(\frac{\vec{u}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \right) \vec{w}_1 \\
&\vdots \\
\vec{w}_i &= \vec{u}_i - \sum_{j=1}^{i-1} \left(\frac{\vec{u}_i \cdot \vec{w}_j}{\vec{w}_j \cdot \vec{w}_j} \right) \vec{w}_j \\
&\vdots \\
\vec{w}_n &= \vec{u}_n - \left(\frac{\vec{u}_n \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \right) \vec{w}_1 - \cdots - \left(\frac{\vec{u}_n \cdot \vec{w}_{n-1}}{\vec{w}_{n-1} \cdot \vec{w}_{n-1}} \right) \vec{w}_{n-1}
\end{aligned}$$

Since all the sides $\vec{w}_i$ are perpendicular, we see that the volume is given by $\prod_{i=1}^{n-1} \|\vec{w}_i\|$. Hence,

$$V = \|\chi(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1})\| = \prod_{i=1}^{n-1} \|\vec{w}_i\| = \prod_{i=1}^{n-1} \left\| \vec{u}_i - \sum_{j=1}^{i-1} \frac{\vec{u}_i \cdot \vec{w}_j}{\vec{w}_j \cdot \vec{w}_j} \vec{w}_j \right\|$$

∎

This theorem reduces to the common case studied in three dimensions:

$$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2.$$

Before we close, we need to make a slight qualification to what we have said in this section. The determinant expression for the cross product is only valid for vector spaces of dimension two or greater. In particular, it does *not* hold for one-dimensional vector spaces, such as $\Re^1$. In one dimension, all bases are of the form $\{\vec{v}\}$ for *any* non-zero vector $\vec{v}$. By normalizing this vector, we obtain a standard basis: $\{\vec{e}\} = \{\vec{v}/\|\vec{v}\|\}$. In this case, the determinant expression given in this section reduces to an 1×1 determinant. This immediately raises some questions because in such a matrix there is only room for one term. But longer speculation convinces one that there is indeed *no* cross product in one-dimensional vector spaces, aside from the trivial one. This observation is obvious geometrically: there is no "room" in one dimensions for a vector perpendicular to any given vector in $\Re$ (besides the zero vector). We can see the same situation algebraically by considering the cross product function $\chi : \Re \mapsto \Re$. The dot product in $\Re$ reduces to multiplication of real numbers, and two vectors in $\Re$ are orthogonal if their product is zero, which is clearly impossible for non-zero real numbers. More generally, let $\vec{v} = x\vec{\imath} \in \Re$; we see that

$$\begin{aligned}
&\chi(\vec{v}) = (ax)\vec{\imath} \text{ for some } a \in \Re \\
&\chi(\vec{v}) \cdot \vec{v} = (ax\vec{\imath}) \cdot (x\vec{\imath}) = ax^2 = 0 \text{ for all } \vec{v} \in \Re \\
&\Rightarrow a = 0 \\
&\Rightarrow \chi(\vec{v}) = 0 \text{ for all } \vec{v} \in \Re
\end{aligned}$$

This is the only possible cross product in $\Re^1$: $\chi(\vec{v}) = 0$ for all $\vec{v} \in \Re^1$. (It is the trivial cross product because the zero vector $\vec{0}$ is orthogonal *(and parallel)* to all other vectors *(and itself)* in $\Re^n$.)

This discussion completes our analysis of the n-dimensional cross product. We can now appreciate a most beautiful result in mathematics: *the* $n \times n$ *determinant gives the cross product in* $\Re^n$.

6 Who needs this? – Applications and Conclusion

> “I exist in the hope that these memoirs, in some manner, I know not how, may find their way to the minds of humanity in Some Dimension, and may stir up a race of rebels who shall refuse to be confined to limited Dimensionality.”
>
> A. Square, in *Flatland* by Edwin A. Abbott [1]

There are literally thousands of applications of the three-dimensional cross product. Classical physics makes constant use of the cross product in analyzing such diverse physical situations as torque, rotation, and magnetism. For a taste of the type of applications that are possible, refer to any calculus-based physics textbook, such as Young and Freedman [9]. The work presented here on the n-dimensional cross product is directly applicable to the physics of four dimensions and higher. In the search for a unified field-theory, there is no doubt that the cross product will play a part.

The three-dimensional cross product gives rise to the *curl* of a three-dimensional vector field. This construct is extremely useful in vector analysis, and plays a prominent role in Maxwell’s equations for electromagnetic phenomena. The n-dimensional cross product similarly gives rise to the curl of an n-dimensional vector field. This result can be applied to Maxwell’s equations to extend their scope to n-dimensional space. Particularly interesting is the two dimensional case of Flatland, where much study has already been conducted [2]. Also of interest is the four-dimensional case of space-time, and the bearing that the dimensionality of a universe has on its electromagnetic phenomena.

Finally, a beautiful result arising from the curl of a three-dimensional vector field is the famous theorem known after its discoverer, George Stokes. Although Stokes’ Theorem is customarily treated only in three dimensions, a number of mathematicians have successfully extended the result to n-dimensional vector fields. This naturally leads them to a definition of the n-dimensional cross product. (See, for example, the work of Michael Spivak [7, pp. 83–85].) Another possible approach to extending Stokes’ Theorem to n-dimensional space is by way of the determinant expression for the n-dimensional cross product derived in this paper.

On this journey of discovery, we have traveled far from the Ordinary Space of Three Dimensions, down to the Curious Flatland of Two Dimensions, up to the Mysterious Land of Four Dimensions, and yet higher still to the Mystical Land of N-Dimensions, and yet lower to the Trivial Land of One Dimension. This journey was made possible by our mathematical wizardry and wit. Along the way, we have discovered a Most Beautiful Truth of Mathematics. We should be proud to have joined with Edwin Abbott “the race of rebels who refuse to be confined to limited Dimensionality.”

7 Appendix

In this appendix, we provide a standard proof of the fact that an $n \times n$ determinant gives the volume of the n-dimensional parallelepiped spanned by its row vectors.

Theorem 13 [Volume of n-dimensional parallelepiped] Let V be the n-dimensional Euclidean space with standard orthonormal basis $B = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$, and let

$$\begin{aligned} \vec{u}_1 &= u_{11}\vec{e}_1 + u_{12}\vec{e}_2 + \cdots + u_{1n}\vec{e}_n \\ \vec{u}_2 &= u_{21}\vec{e}_1 + u_{22}\vec{e}_2 + \cdots + u_{2n}\vec{e}_n \\ &\vdots \\ \vec{u}_n &= u_{n1}\vec{e}_1 + u_{n2}\vec{e}_2 + \cdots + u_{nn}\vec{e}_n \end{aligned}$$

be n vectors in V. Then the volume of the parallelepiped spanned by $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is given by the absolute value of $n \times n$ determinant

$$\begin{vmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{vmatrix}.$$

Proof: Apply the Gram-Schmidt orthogonalization process to the set of vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ to obtain a new set of *orthogonal* vectors $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ that span the same subspace of V. By not modifying the lengths of the vectors, but only making the set orthogonal, we see that the volume of the parallelepiped spanned by $\{\vec{w}_i\}$ is equal to the volume of the original parallelepiped spanned by $\{\vec{u}_i\}$. To be more precise, let

$$\begin{aligned} \vec{w}_1 &= \vec{u}_1 \\ \vec{w}_2 &= \vec{u}_2 - \left(\frac{\vec{u}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\right)\vec{w}_1 \\ &\vdots \\ \vec{w}_i &= \vec{u}_i - \sum_{j=1}^{i-1}\left(\frac{\vec{u}_i \cdot \vec{w}_j}{\vec{w}_j \cdot \vec{w}_j}\right)\vec{w}_j \\ &\vdots \\ \vec{w}_n &= \vec{u}_n - \left(\frac{\vec{u}_n \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\right)\vec{w}_1 - \cdots - \left(\frac{\vec{u}_n \cdot \vec{w}_{n-1}}{\vec{w}_{n-1} \cdot \vec{w}_{n-1}}\right)\vec{w}_{n-1} \end{aligned}$$

We see that this process involves only adding scalar multiples of one row to another, and from linear algebra we know that when such an operation is applied to a matrix, its determinant remains unchanged. We thus see that

$$V = Volume(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n) = \begin{vmatrix} \longleftarrow \vec{u}_1 \longrightarrow \\ \longleftarrow \vec{u}_2 \longrightarrow \\ \vdots \\ \longleftarrow \vec{u}_n \longrightarrow \end{vmatrix} = \begin{vmatrix} \longleftarrow \vec{w}_1 \longrightarrow \\ \longleftarrow \vec{w}_2 \longrightarrow \\ \vdots \\ \longleftarrow \vec{w}_n \longrightarrow \end{vmatrix}.$$

$$\text{Let } W = \begin{bmatrix} \longleftarrow \vec{w}_1 \longrightarrow \\ \longleftarrow \vec{w}_2 \longrightarrow \\ \vdots \\ \longleftarrow \vec{w}_n \longrightarrow \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix}.$$

Then, $V = \det W$. Note that

$$\begin{aligned} WW^T &= \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix} \begin{bmatrix} w_{11} & w_{21} & \cdots & w_{n1} \\ w_{12} & w_{22} & \cdots & w_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ w_{1n} & w_{2n} & \cdots & w_{nn} \end{bmatrix} \\ &= \begin{bmatrix} \vec{w}_1 \cdot \vec{w}_1 & \vec{w}_1 \cdot \vec{w}_2 & \cdots & \vec{w}_1 \cdot \vec{w}_n \\ \vec{w}_2 \cdot \vec{w}_1 & \vec{w}_2 \cdot \vec{w}_2 & \cdots & \vec{w}_2 \cdot \vec{w}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{w}_n \cdot \vec{w}_1 & \vec{w}_n \cdot \vec{w}_2 & \cdots & \vec{w}_n \cdot \vec{w}_n \end{bmatrix} \\ &= \begin{bmatrix} \|\vec{w}_1\|^2 & 0 & \cdots & 0 \\ 0 & \|\vec{w}_2\|^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|\vec{w}_n\|^2 \end{bmatrix}. \end{aligned}$$

The last two equalities follow because, since $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ is orthogonal by construction,

$$\langle \vec{w}_i | \vec{w}_j \rangle = 0 \text{ for } i \neq j$$

$$\text{and } \langle \vec{w}_i | \vec{w}_i \rangle = \|\vec{w}_i\|^2.$$

Thus,

$$\begin{aligned} \det(WW^T) &= \|\vec{w}_1\|^2 \|\vec{w}_2\|^2 \cdots \|\vec{w}_n\|^2 \\ &= (\|\vec{w}_1\| \|\vec{w}_2\| \cdots \|\vec{w}_n\|)^2 \\ &= V^2. \end{aligned}$$

The first equality follows because the determinant of a diagonal matrix is the product of the diagonal entries. The last equality follows because the volume of a parallelepiped with *perpendicular* sides is the product of the lengths of the sides. Also,

$$\begin{aligned}\det(WW^T) &= \det(W)\det(W^T)\\ &= \det(W)\det(W)\\ &= \det(W)^2.\end{aligned}$$

Thus, combining the last two arguments, we see that

$$\det(W)^2 = V^2 \Longleftrightarrow |\det W| = V.$$

∎

8 References

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